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## Taking a leaf from experts on effective instructional approaches related to the teaching of optimization problems in calculus

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*This study used a qualitative questionnaire to elicit experts' views regarding effective instructional approaches related to the teaching of optimization problems in first-semester calculus. A thematic analysis was conducted on responses from 11 instructors who completed the questionnaire. It revealed that all the instructors considered the construction and use of diagrams, providing students with a set of steps to follow, and creating ample individual and group opportunities (during course lectures and outside the classroom) to be helpful in supporting students' learning about these types of problems. The experts further identified several technologies, including Desmos, GeoGebra, and WebAssign visuals, they consider to be helpful in supporting students' learning about optimization problems. Additionally, the experts provided helpful insights regarding calculus textbooks' strengths and weaknesses in their presentation of optimization problems. Implications for different stakeholders, including calculus instructors and textbook selection committees in mathematics departments, are discussed.*

### 1. Introduction

Optimization problems not only form an integral part of first-semester calculus, commonly referred to as Calculus I in the United States, but they are also a type of problems that are well known to be particularly challenging for students to solve (cf. LaRue & Infante, 2015). In fact, there is a substantial and growing body of research that has reported on students' difficulties when working with optimization problems (cf. Borgen & Manu, 2002; Brijlall & Ndlovu, 2013; Heid, 1988; Klymchuk et al., 2010; Maharaj, 2013; Mkhathswa, 2016, 2019; Swanagan, 2012; Ubuz & Ersoy, 1997; Villegas et al., 2009; White & Mitchelmore, 1996). Thus, the current study aims to create a resource of teaching strategies. These strategies are designed to address previously reported students' difficulties when working optimization problems. To accomplish this, the current study examined veteran calculus instructors' (hereafter, experts) views on effective instructional approaches related to the teaching of optimization problems in Calculus I.

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Calculus I is typically taught by instructors with a wide spectrum of teaching experience, including graduate students who often serve as teaching assistants, early career and tenure-track (probationary) professors, and late career and tenured (permanent) professors in the United States. It is my hope that the list of effective teaching approaches presented in this study will be especially helpful for any instructor with a limited or no calculus teaching experience at all, regardless of the stage of their teaching career, namely early, middle, or late career stage, respectively. After all, evidence from research shows that instructors tend to look to research for help on how to best teach certain concepts/topics in their disciplines. (cf. Drill et al., 2013).

Advances in technology have led to a rise in the invention and usage of various educational technologies in the teaching of many undergraduate mathematics courses in the United States, including Calculus I. Furthermore, a number of studies have reported on instructors' propensity to closely follow the presentation of content in mathematics textbooks during course lectures (cf. Mkhathshwa, 2016; Reys et al., 2004). Thus, in addition to examining experts' views regarding best practices in the teaching of optimization problems, the present study investigated experts' perceptions regarding the role of technology and calculus textbooks in the teaching of optimization problems in Calculus I. In essence, the present study seeks to address the following research questions:

1. What do veteran calculus instructors consider to be effective instructional strategies in the teaching of optimization problems in Calculus I?
2. What teaching technologies do experienced calculus instructors use in their teaching of optimization problems in Calculus I?
3. What do calculus instructors identify as weaknesses and strengths in the presentation of optimization problems in Calculus I textbooks?

## **2. Related Literature**

### **2.1 Students' difficulties when working with optimization problems**

While the main focus of the present study is on effective instructional approaches related to the teaching of optimization problems in Calculus I, it is worthwhile to review existing literature on students' difficulties when working with optimization problems for comparison.

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Specifically, this will help determine the extent, if possible, to which the teaching strategies presented in this study could potentially address (or fail to address) students' difficulties with optimization problems previously reported in the research literature.

A common theme that emerges from research that has examined students' thinking about optimization problems in Calculus I is that formulating the objective function i.e., the function whose minimum/maximum value is to be found is generally problematic for students (cf. Klymchuk et al., 2010; Mkhatshwa, 2019; Swanagan, 2012; Villegas et al., 2009; White & Mitchelmore, 1996). In fact, LaRue and Infante (2015) observed that this is problematic even when students are tasked with solving routine optimization problems where constructing the objective function is simple, and observation also made by Klymchuk et al. (2010). Another common finding is that students often struggle with finding critical numbers or extreme values of the objective function, even after successfully formulating the function (cf. Brijlall & Ndlovu, 2013; Dominguez, 2010; Mkhatshwa, 2019; Swanagan, 2012). As a matter of fact, findings by Swanagan (2010) indicate that sometimes students rely on guesswork to determine an objective function's critical numbers or extreme values. Lastly, several researchers have found that verifying/justifying extrema using formal methods such as the first or second derivative test is particularly challenging for many students when working with optimization problems in Calculus I (cf. Borgen & Manu, 2002; Mkhatshwa, 2019). Part of this challenge stems from not being able to recall these formal methods or applying them incorrectly.

## **2.2 The role of technology in the teaching of optimization problems**

One of the items in the questionnaire used in this study specifically asked experts about the technologies they use and how they use these technologies in their teaching of optimization problems. Consequently, this section provides a survey of literature on the use of technology in the teaching and learning of optimization problems. To date, only two studies have reported on the use of technology in the teaching or assessment of students' learning about optimization problems in calculus. Çekmez (2021) reported on a control group whose exposure to optimization problems was limited to a traditional lecture method and an experimental group whose exposure to optimization problem involved the use of GeoGebra, a dynamic mathematics software. Analysis of students' performance in an optimization

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problem they were asked to solve revealed that students in the experimental group outperformed their counterparts in the control group. Furthermore, students in the experimental group demonstrated superior understandings of the quantities and relationships among the quantities involved in the optimization problem compared to their counterparts in the control group. Arango et al. (2015) reported on a successful incorporation of GeoGebra in the teaching of optimization problems in calculus. These researchers remarked that the use of GeoGebra complements paper-based methods of solving optimization problems in that it affords students the ability to visualize the critical number(s) and extreme value(s) of the objective function. Additionally, results from Arango et al.'s (2015) study suggest that GeoGebra is a useful tool that could be used by students to, among other things, verify the correctness of their paper-based solutions when working with optimization problems in calculus.

### **2.3 The role of textbooks in the teaching and learning of mathematics**

While the principal aim of the current study is to explore, through the lens of experts' teaching experiences of Calculus I, effective instructional approaches related to optimization problems, it is crucial to examine the role of textbooks in the teaching and learning of mathematics in general. Most importantly, one of the items in the questionnaire used in the present study elicited experts' perceptions regarding strengths and weaknesses in the presentation of optimization problems by calculus textbooks. Although much research has reported on opportunities to learn provided in K-12 mathematics textbooks, and only a few studies have examined similar opportunities provided by undergraduate mathematics textbooks. Until now, only two studies (Mkhatshwa, 2023; Mkhatshwa & Doerr, 2016) have reported on the opportunity to learn about optimization problems provided by undergraduate calculus textbooks in the United States. Findings of these studies indicate that calculus textbooks in the United States are dominated by tasks with a camouflage context, tasks with matching information, and reproduction tasks i.e., low cognitive demand tasks.

A common finding of research that has examined the relationship between opportunities to learn provided by classroom instruction and mathematics textbooks at both the K-12 level and at the university level is that the presentation of content, including optimization problems in calculus, during courses lectures tend to closely follow the presentation of similar content in mathematics textbooks (cf. Begle, 1973; Mkhatshwa,

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2016; Reys et al., 2004; Wijaya et al., 2015). In fact, Reys et al. (2004) eloquently argued that “...the choice of textbooks often determines what teachers will teach, how they will teach it, and how their students will learn” (p. 61), a sentiment that has been shared by other scholars (cf. Begle, 1973). Motivated by the importance of textbooks in students’ learning about mathematics, the present study seeks to document experts’ perspectives regarding opportunities to learn about optimization problems provided by calculus textbooks in the United States.

### **3. Methods**

#### **3.1 The questionnaire: Design and validation**

This qualitative study used an eleven-item online Qualtrics questionnaire (please see Appendix A) that was administered in the fall semester of 2022 to examine experts’ views on the teaching of optimization problems in calculus I, among other things. The principal objective of the study was to investigate effective ideas and strategies related to the teaching of optimization problems in Calculus I. Other than Questions 1 and 2 in the questionnaire that were included to elicit experts’ experiences with teaching Calculus I and the nature (in terms of research productivity) of the academic institutions they were affiliated with at the time of completing the questionnaire, the rest of the questions in the questionnaire were for the most part informed by findings of existing literature on students’ thinking about optimization problems at the level of Calculus I.

The goal for designing the questionnaire was six-fold. First, to gain insight on what calculus instructors consider to be easy/challenging for students when working with optimization problems (Items 3 and 4 in the questionnaire). Second, to gain insight on what calculus instructors consider to be effective instructional approaches in the teaching of optimization problems (Item 5 in the questionnaire). Third, to gain insight on calculus instructors’ perceptions regarding the role of technology, or other instructional tools, in the teaching of optimization problems (Item 6 and 7 in the questionnaire). Fourth, to gain insight on frequently used real-world contexts, if any, in the teaching of optimization problems (Item 8 in the questionnaire). Fifth, to gain insight on the range (wide or limited) of textbooks used by experts in the teaching of optimization problems (Item 9 in the questionnaire). Sixth, to gain insight on calculus instructors’ views regarding strengths and weaknesses in the presentation of optimization problems in calculus textbooks (Items 10 and 11 in the questionnaire).

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Three subject experts were invited to comment on two aspects of validity, namely face validity and content validity, as it relates to the questionnaire used in the current study. I remark that these experts are not among the 11 experts who are participants in the study. They all had sufficient experience teaching and tutoring students taking Calculus I in the United States. In using the term face validity, I am refereeing to the suitability of the questionnaire in addressing the research questions investigated in the study (Martinez, 2017). Furthermore, in using the term content validity, I refer to the questionnaire's ability to capture all aspects related to the teaching and learning of optimization problems in calculus (Martinez, 2017). In their independent evaluations of the questionnaire, the three experts deemed the questionnaire to be valid in terms of the two aspects of validity that were assessed i.e., face validity and content validity.

### **3.2 Sampling method and participants**

Using convenience sampling, 11 experts (herein denoted as E1 through E11) were recruited to participate in the study reported in this paper. The recruitment of the participants was accomplished in two phases. In the first phase, I sent out invitation emails to calculus instructors I know in the United States. One of the experts was my colleague at the same institution I was working for at the time of the study, another expert was from an institution I was previously affiliated with, some experts are colleagues I met at academic conferences, and yet other experts are colleagues in the teaching profession that I do not know at a personal level, but I am familiar with work they had done related to the teaching and learning of calculus through their scholarly works. In the second phase, I searched the internet using key words such as "calculus coordinator" or "calculus director." I then sent out an invitation emails to participate in the study to anyone whose contact information (email) was listed as a calculus coordinator/director in a mathematics department in the United States.

In total, I sent out a total of 14 invitations. Of these invitations, 11 experts agreed to participate in the study. Only two of the 11 participants partially completed the questionnaire with an 82% completion rate-the rest of the participants completed all the items included in the questionnaire. Three participants had taught 1-5 sections, two participants had taught 6-10 sections, one participant had taught 11-15 sections, one participant had taught 16-20 sections, and four participants had taught over 20 sections of Calculus I sections, respectively. Furthermore, eight participants were

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affiliated with R1 institutions i.e., institutions with a very high research activity, two participants were affiliated with R2 institutions i.e., institutions with a high research activity, and one participant was affiliated with liberal arts college i.e., an institution with a moderate research activity.

### 3.3 Data analysis

Thematic analysis was used to analyze data for the study i.e., experts' responses to Items 3 through 11 in the questionnaire. According to Braun and Clark (2006), thematic analysis is "...a method for identifying, analysing and reporting patterns (themes) within data" (p. 79). The term theme, as used in this study, refers to a common response to the questionnaire items given by at least two experts. For instance, if one expert noted cost, another expert mentioned revenue, and another expert remarked about profit in response to Item 8 in the questionnaire that elicited experts' perceptions regarding typical real-world contexts the experts use in their teaching of optimization problems in Calculus I, I identified the economics context as a theme.

## 4. Results

### 4.1 Students' strengths and weaknesses when working with optimization problems

*Performing algebraic manipulations and solving simple problems:* In response to Item 3 in the questionnaire that was used to elicit experts' views regarding students' strengths when working with optimization problems, nine of the 11 experts interviewed in this study remarked that performing algebraic manipulations (e.g., calculating derivatives) and solving simple problems is often straightforward for most students. The following is a reproduction of exemplary responses:

Students have found the process of differentiating an optimizing function [the function whose maximum or minimum value is sought] straightforward. At that point, they are pretty good at differentiating so doing the calculus of the optimization problem isn't difficult for them (E4).

Performing algebraic manipulations (e.g., differentiation), especially once they have found the relevant equations in a given scenario (E5).

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Taking the derivative [of the optimizing function], Setting the derivative equation to zero, and solving. These steps are always the same for any optimization problem, and students have a lot of experience with finding derivatives by then (E8).

Farmer-type problems (i.e., find the largest-area pen that you can build with this amount of fence). Also, straight numerical problems, like "find the absolute min and max of this function on this closed interval." (E2).

The very simple problems involving area of a rectangle/perimeter of a rectangle (E7).

A careful examination of E4, E5, and E8's responses suggest that calculating derivatives, once the objective function (also known as the optimizing function) has been determined (if not given), is easy for most students when working with optimization problems. Taken together, E2 and E7's responses suggest that solving area maximization/minimization problems involving rectangles is generally well understood by students. It should be noted that although E2 did not specifically mention the word "rectangle" in his response, farmer-type problems are common and standard problems that are covered in most, if not all, calculus classes or textbooks in the United States. Furthermore, farmer-type problems generally involve working with rectangles (cf. LaRue & Infante, 2015).

*Mathematizing word problems:* In response to Item 4 in the questionnaire that was used to investigate experts' perspectives regarding students' difficulties when working optimization problems, all the experts made remarks that suggested that translating prose to text i.e., mathematizing (Freudenthal, 1993) these type of problems is a common challenge for many students. The following is a reproduction of a few exemplary responses given by some of the experts:

Most of it. :-) 1.) Coordinating all of the pieces in a big problem is hard for students. They often get lost in the details and forget the big ideas. 2.) Setting up the optimization problem is often hard for students (although for some reason, I think my students did pretty well with this this semester). 3.) Algebra can also be tricky. They know to set the derivative equal to 0, but they don't always know

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how to solve for  $x$  (algebraically). 4.) They often forget to use the first or second derivative test to determine if their critical value really gives the optimal value. I'm not sure they realize why this is a big deal (E1).

Word problems that aren't farmer problems, or that require understanding of the physical situation to set up the objective function [synonym for optimizing function], or where the feasible interval isn't bounded, or where the objective function involves angles. Examples include things like: A train starts at P1 and is traveling north at  $\alpha$  mph, and a truck starts at P2 and is traveling east at  $\beta$  mph. Or, my favorite: Given a circular metal disk of radius 10, you want to cut one sector of angle  $\theta$  and weld the edges together to make a cone. What angle  $\theta$  gives the cone of largest volume (E2)?

One of the challenging parts of optimization problems for students seems setting up the initial variables and translating given problem into math problems. The other challenging part seems explaining logically why at certain position it could have the optimized results using either the first derivative or second derivative test. A lot of students forget that part, and simply assuming that the optimization happen at any critical number they found (E10).

E1, E2, and E10 all commented on students' difficulties associated with setting up the optimizing function when working with word problems. However, only E1 and E10 noted that verifying extrema i.e., using the first or second derivative test to check if the critical value(s) they found would result in the optimal value(s) for the optimizing function is generally problematic for calculus students. Additionally, E1 and E7 are the only experts who remarked that sometimes students' difficulties with solving optimization problems stem from lack of facility with working with complex algebra, especially when trigonometric functions are involved.

#### **4.2 Effective ways to help students become proficient at solving optimization problems**

*Encourage students to construct and use diagrams:* In response to Item 5 in the questionnaire that prompted experts to comment on effective

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ways they have used to support students' learning about optimization problems, three experts emphasized the importance of drawing and using diagrams that serve as visual illustrations of the scenarios described in these problems. The following is a reproduction of these experts' responses to Item 5 in the questionnaire:

I always tell students to draw a diagram of the object/scenario they wish to optimize and label appropriate quantities. This can help them to realize the formula for the appropriate geometric shape. (E4).

For interpreting problems, I have often found it useful to encourage students to draw and label diagrams whenever possible to help them visualize the relationships (E5).

I always encourage them helping each other, drawing pictures, and leveraging all of their resources that they have online or in-person (E6).

Although E6 did not provide a rationale for encouraging students to draw diagrams when solving optimization problems, from E4 and E5's responses one could infer that the purpose of drawing and using diagrams is to help students make sense of the quantities and relationships among the quantities involved in these problems. In other words, the purpose of these diagrams is to support students' quantitative reasoning while working on these type of problems (cf. Smith III & Thompson, (2007; Thompson, 1993; Thompson, 2011).

*Provide a set of steps students could follow:* In response to Item 5 in the questionnaire, five experts made remarks to the effect that they provide students with a set of steps students could follow when solving optimization problems. The following is a reproduction of some exemplary responses to the aforementioned questionnaire item:

We often talk through the big ideas (generally) without doing all of the steps. I also give them several problems to solve (many of which are graded). (E1).

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I give them a step by step process, that involves writing the objective function (including realizing that they are wrong if their function has more than one variable), defining the feasible interval, finding critical points, and proceeding from there (E2).

We focus first on what quantity we are trying to maximize or minimize. We set up an equation to represent that quantity, and then we ask ourselves if the equation we have written down describes our exact problem (for example, if we know our rectangle is supposed to have an area of  $400\text{cm}^2$ , then is that information incorporated into our function?) We do several problems where they \*only\* set up the function. This allows for more practice and experience setting up the [optimizing] function (E7).

I tell them that the construction of objective function proceeds in two steps: (1) finding a formula for the quantity to be maximized/minimized; often it is a known geometric formula that should be memorized. (2) reducing the number of variables to 1, if there are more than 1. This requires one to express the relation between different variables as an equation, and to use it to eliminate one of the variables (E9).

A close examination of the preceding responses reveals that while E1 and E2 provide a set of steps for solving optimization problems in general, E7 and E9's responses suggest that these experts provide steps that are specific to constructing the optimizing function. Indeed, constructing the optimizing function is particularly challenging for most calculus students when working with optimization problems based on the evidence provided in Section 4.1.

*Individual and group practice problems:* In response to Item 5 in the questionnaire, three experts noted that they provide numerous individual and group opportunities for students to practice working with optimization problems. The following is a reproduction of these experts' responses to Item 5 in the questionnaire:

We often talk through the big ideas (generally) without doing all of the steps. I also give them several problems to solve (many of which are graded). (E1).

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It is important to show the students that they are capable of doing these problems on their own. Thus, I tend to give them a lot of "easy" optimization problems to try on their own first before I move on to more difficult examples. I also give them plenty of opportunities for practice. This includes in-class and outside-of-class worksheets (E8).

Examples, group quizzes in class (E11).

Arguably, the use of graded assignments on optimization problems as E1 does, encourages students to put in maximum effort when working on these assignments and not just to complete them for the sake of completing an assignment. While E1 was not specific whether these assignments are completed in the classroom or outside the classroom, E8 specifically noted that the opportunities he provides takes place both during classroom instruction and outside the classroom. Unlike E1 and E8 who were not specific on whether the assignments they provide are to be completed individually or in groups, E10 specifically mentioned that he gives students group quizzes during classroom instruction. Because effective instructional approaches are the central focus of this paper, I have reproduced all the experts' responses to Item 5 in the questionnaire in Appendix B. Readers will observe that although the suggested teaching approaches in the aforementioned appendix are not necessarily themes, they are indeed worthy of consideration when teaching optimization problems, especially for novice calculus instructors.

### **4.3 Educational technologies and tools in the teaching of optimization problems**

In response to Item 6 in the questionnaire, which I used to elicit experts' views regarding technological tools they use to support students' learning about optimization problems in Calculus I, seven experts indicated that they use several educational technologies, including Desmos, GeoGebra, WebAssign visuals, or a combination of these technological tools. The following is a reproduction of five exemplary responses to Item 6:

WebAssign and Crowdmark for HW assignments. Sometimes (rarely) we will graph the optimizing function and talk about what values mean (E1).

I try to use visuals from GeoGebra and WebAssign (E3).

I call on Desmos for technology in optimization. The optimization functions are often difficult to visualize, so use of Desmos can enable students to verify their result graphical (e.g. is the answer they got truly a local minimum/maximum). (E4).

I often use graphing tools like desmos and GeoGebra to help students generate graphs. Geogebra is especially very good because you can create animations and moving sliders that students can adjust to see various relationships and how variables change in real-time (E5).

Sometimes we use graphing utilities like Desmos to graph the function and look at the maximum/minimum to see if it matches what we found algebraically. It's also helpful with the rare problems where there is more than one critical point and we have to figure out which one gives the actual absolute maximum or minimum in the specific domain (E7).

While experts like E1 and E3 only mentioned technological tools they use to support students' learning about optimization problems without discussing the rationales or reasons for using these technologies, experts like E4, E5, and E7 provided detailed responses. E4, for example, uses Desmos to help students graph and visualize optimizing functions. Furthermore, E4 further remarked that the "...use of Desmos can enable students to verify their result graphically (e.g. is the answer they got truly a local minimum/maximum)?" It should be noted that three other experts expressed that they do not use any technological tool in their teaching of optimization problems. In fact, one of these experts expressed interest in learning to use technology in teaching of optimization problems as can be seen in the following excerpt: "I currently do not use technologies to teach optimization problems. But I would be interested in learning how one could supplement their teaching of optimization problems with technology." (E8). I thus hope that the information presented in this section regarding various technologies used by other experts, how or why they use these technologies will serve as a great starting point for experts like E8 who are interested in

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learning about integrating technological tools in their teaching of optimization problems in calculus.

In response to Item 7 in the questionnaire, which I used to examine experts' views regarding non-technological tools they use to support students' learning about optimization problems in Calculus I, three experts mentioned providing ample opportunities for students to work with optimization problems in and outside the classroom. These experts' responses are in many ways similar to those discussed in Section 4.2. Consequently, I chose not to reproduce them in this section. Instead, I reproduced all the experts' responses to this item in Appendix C. I further note that, although there are no other themes (besides providing ample practice problems) in the experts' responses to Item 7 in the questionnaire (Appendix C), most of the responses offer valuable insights on effective teaching approaches related to the teaching of optimization problems in Calculus I. As can be seen in Appendix C, four experts provided responses that suggest that they do not use any special non-technological tools in their teaching of optimization problems in Calculus I.

#### **4.4 Typical real-world contexts used by experts in the teaching of optimization problems**

In response to Item 8 in the questionnaire, which I used to examine experts' views regarding typical real-world contexts they use in their teaching of optimization problems in Calculus I, seven experts noted that they often use economics and kinematics contexts, respectively. The following is a reproduction of exemplary responses to Item 8 in the questionnaire:

Economics (minimizing cost) kinematics (?) (We have problems where we minimize the time to get to a point, or finding the maximum acceleration, etc.) perimeter/area to maximize space or minimize perimeter I'm sure I'm missing several. Most of these seem to be "real-world" contexts (E1).

We do use economics, but since I teach the Calc 1 class for physical science and engineering, most of the examples are physical. They involve finding things like the least cost road that can be built with some combination of bridge and surface roadway, or the shortest board that can be placed with some geometric constraint, or the

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minimum distance between two moving objects, etc. We have a separate calc 1 class that is more oriented toward business, econ, and social science, and that's not the one I usually teach (E2).

While the question mark (?) in front of the word kinematics in E1's response may be taken to mean that this expert was not sure about whether or not she uses optimization problems that would qualify as having a kinematics context in her teaching of Calculus I, her claim that some of the problems she uses involve finding the maximum acceleration provides evidence of using a kinematics context. A kinematics context, in my understanding is one that involves the concepts of distance, position, velocity, speed, and acceleration. E2's claim that most of the examples he uses in his teaching of Calculus 1 are physical such as finding the "...minimum distance between two moving objects" provides evidence of using kinematics contexts in his teaching of optimization problems.

#### **4.5 Strengths and weaknesses in the presentation of optimization problems in calculus textbooks**

Before delving into the experts' views regarding strengths and weaknesses in how calculus textbooks they have used present optimization problems, I would like to note that nine of the 11 textbooks used by the 11 experts in this study were authored by James Stewart, a well-known calculus textbook author in the United States. One of the other two textbooks was authored by Matthew Boelkins while the other textbook was authored by three authors, namely Joel Hass, Christopher Heil, and Maurice Weir. The aforementioned summary on the calculus textbooks used by the experts who participated in this study is based on information obtained from the experts' responses to Item 9 in the questionnaire, which I used to gather details about calculus textbooks used by the experts in their teaching of optimization problems in Calculus I. The details include textbook names, authors, and edition, respectively.

In response to Item 10 in the questionnaire, which I used to examine experts' views regarding weaknesses in the presentation of optimization problems in calculus textbooks, three experts expressed disappointment that the examples or practice problems provided in calculus textbooks are not realistic. The following is a reproduction of the three experts' responses to Item 10 in the questionnaire:

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The main issue I see is that some books get a little too exotic with the geometry, or use physics examples that are too abstract (e.g., luminance, electromagnetism). (E2).

There are rather standard problems that students do not view as real-life applications (E3).

Most "real-life" optimization problems are real "fake real-life" optimization problems. For example, no one is going to calculate how far one needs to run before swimming to an island to optimize the time spent on traveling (E8).

E2 noted that some of the optimization examples given in calculus textbooks are abstract, while E3 noted that optimization problems provided in calculus textbooks are generally not viewed by students "as real-life applications", perhaps because they are often abstract or have a camouflage context. E8 not only remarked that optimization problems in calculus textbooks have unrealistic contexts, but also went on to provide an example of one such problem i.e., calculating how far one needs to run before swimming to an island to optimize the time spent on traveling. Two other experts lamented that solutions to most optimization examples and practice problems provided in calculus textbooks are easily accessible to students via the internet. In response to Item 10 in the questionnaire, E11 commented, "Solutions to HW [homework] are everywhere online..." (E11). In response to the same item, another expert remarked, "many of the problems have been recycled over years and years, and as such the solutions are readily available to students on external resources such as Chegg [An American education company that provides many services, including step by step solutions to practice problems]" (E4). Remarking on a creative approach to alleviate the severity of this weakness in calculus textbooks' presentation of optimization problems, E4 added, "this forces me and other instructors to write novel questions on optimization (e.g. cats and their play pens)." Two other experts stated that they could not think of any weakness regarding the presentation of optimization problems in calculus textbooks they have used.

In response to Item 11 in the questionnaire, which I used to examine experts' views regarding strengths in the presentation of optimization problems in calculus textbooks, five experts expressed

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appreciation for the abundance of optimization examples and practice problems provided in calculus textbooks. The following is a reproduction of three exemplary responses to Item 11 in the questionnaire:

I like how, active Calculus specifically, uses relatable real-life situations in presenting optimization problems. The text also provides some questions that students explore before they encounter the concept of optimization. Many texts have several solved examples on optimization problems and several exercises for students to try. This, I think, is a strength (E5).

Lots of examples for students to see (E6).

Many examples and problems for students to try on their own (E8).

While E6 only remarked on the abundance of optimization examples as a strength for calculus textbooks, E5 and E8 remarked on the abundance of optimization examples and practice problems, respectively. Two other experts did not respond to this item, perhaps because they could not think of anything in particular they could attribute as a strength in how calculus textbooks present optimization problems. Although not a theme, one expert noted that there was nothing he could note as a strength: "Most are pretty standard, and nothing stands out as a strength." (E2). E2 elaborated, "reading about this topic [optimization problems] is very dry for students, so there's no substitute for covering it in a class." Another expert expressed appreciation for the graphics used by calculus textbooks in their presentation of optimization problems as can be seen in her response: "There are usually some good graphics, and they include the graph of the [optimizing] function a lot of the time." (E7).

## 5. Discussion and Conclusions

In what follows, I discuss the main findings of the study. First, based on the experiences of a majority of the experts in this study, performing algebraic manipulations such as calculating derivatives of optimizing functions and finding critical values of these functions is generally well understood by students. On the contrary, several studies have reported that finding critical numbers is sometimes challenging for calculus students (cf. Brijlall & Ndlovu, 2013; Dominguez, 2010; Mkhathshwa, 2019; Swanagan,

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2012). The experts further noted that solving application problems involving optimization problems is typically difficult for students. Particular difficulties exhibited by students when solving application problems include but are not limited to formulating the optimizing function and verifying extrema (i.e., showing, using the first or second derivative test, to show that a particular critical value yields the optimal value for the optimizing function) is particularly challenging for most calculus students. The experts' observations regarding students' difficulties when tasked with solving optimization problems are consistent with findings of previous research on students' thinking about optimization problems in Calculus I (cf. Borgen & Manu, 2002; Klymchuk et al., 2010; Mkhathshwa, 2019; Swanagan, 2012; Villegas et al., 2009; White & Mitchelmore, 1996).

Second, all the experts in this study remarked on several effective teaching approaches they have used to support students' learning about optimization problems in Calculus 1. These approaches include, but are not limited to *encouraging students to construct and use diagrams (when it makes sense) when working with optimization problems, providing a set of steps that could serve as a guideline for students when solving optimization problems, and creating ample individual and group opportunities for students to practice solving a wide range of optimization problems during course lectures as well as outside the classroom*. A complete list of the teaching approaches used by the 11 experts are reproduced in Appendices B and C. Additionally, most experts endorsed the use of educational technologies like Desmos, GeoGebra, and WebAssign to facilitate understanding. They highlighted how these tools can aid in visualizing optimizing functions and verifying solutions. The interested reader is encouraged to consider the work of Arango et al. (2015) and Çekmez (2021) as examples on how one of these technologies (GeoGebra) could be used in the teaching of optimization problems in calculus. As an experienced calculus instructor myself, I strongly recommend adopting most if not all of the suggested teaching strategies and technologies suggested by the experts in the teaching of optimization problems and other concepts such as the derivative in Calculus I. While these teaching strategies and technological tools could work for nearly any calculus instructor, they can be especially helpful for novice calculus instructors such as graduate teaching assistants in mathematics departments.

Third, it is commendable that most of the experts noted using other another real-world context (economics), in addition to the kinematics

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context in their teaching of optimization problems in Calculus I. Findings of previous research on students' thinking about concepts (e.g., derivatives) directly related to optimization problems indicate that such concepts are typically well understood in kinematics contexts and that projecting these concepts to other real-world non-kinematics contexts is generally difficult for students (cf. Ibrahim & Rebello, 2012). In fact, recent research (Mkhatshwa, 2022) on examples and practice problems on derivatives provided by one of the commonly used calculus textbooks in the United States indicate that these examples and practice problems are dominated by a kinematics context. Additionally, several of the experts in this study expressed disappointment that a majority of the examples and practice problems on optimization provided in calculus textbooks they have used have a camouflage context, and consequently are not interesting to students. Some experts further noted that these examples and practice problems are fairly standard and found in most calculus textbooks. Consequently, solutions to these examples and practice problems are easily accessible to students online via platforms such as Chegg. On a positive note, nearly half of the experts in this study expressed appreciation for the multitude of optimization examples and practice problems provided in most calculus textbooks. To this end, I recommend that calculus instructors and textbook selection committees in mathematics departments consider several things when adopting/selecting calculus textbooks for their courses. Examples of things to consider include opportunities to work with optimization problems (and other topics in calculus in general) that have several real-world contexts, opportunities to work with a variety of examples and practice problems (i.e., standard and non-standard), and opportunities to work with examples and practice problems that are more realistic in nature to promote interest to learn about these problems among students.

## **6. Study Limitations and Potential Areas for Future Research**

I would like to note that while the results reported in this study are quite informative and potentially helpful for many calculus instructors, especially novice instructors such as graduate teaching assistants in mathematics departments, the fact that the experts in this study were not interviewed represents a limitation. Specifically, conducting structured interviews could have provided opportunities for the experts to elaborate in greater detail regarding how they use certain strategies or technologies to

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support students' learning about optimization problems. This could be a subject for future research. To alleviate this concern, a majority of the experts provided sufficient details in their responses to the questionnaire items, especially to the items about effective teaching strategies and technologies related to the teaching of optimization problems in Calculus I. Furthermore, the interested reader is specifically referred to, in the Discussion and Conclusions section of the paper, to consider the works of several scholars (cf. Arango et al. 2015; Çekmez (2021) for examples on how to incorporate one of the technologies (GeoGebra) used by several experts in this study to support students' learning about optimization problems.

The data reported in the current study was analyzed by one researcher. Consequently, it was not possible to measure the inter-rater reliability of the questionnaire used in the study. I recommend that future research that uses the questionnaire used in the current study (or its adaptation thereof) also measures the inter-rater reliability of the questionnaire, in addition to assessing the face and content validity of the same instrument. I strongly recommend that future research examines the effectiveness (or lack thereof) of the teaching strategies documented in the current study in different instructional settings or with different student populations. Lastly, I remark that the fact that the sample size (11 participants) is small and that the survey respondents were not randomly selected limits the extent to which the results of the study could be externally valid i.e., generalized to other settings.

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### Appendix A-Questionnaire

1. According to the Carnegie Classification System, what is the rank (e.g., R1) of the institution you are affiliated with?
2. How many sections of Calculus I have you taught?
 

|        |         |          |          |         |
|--------|---------|----------|----------|---------|
| 1 to 5 | 6 to 10 | 11 to 15 | 16 to 20 | over 20 |
|--------|---------|----------|----------|---------|
3. Drawing on your experience teaching calculus I, what have you found to be easy/straightforward for students when working with optimization problems? Explain.
4. Drawing on your experience teaching calculus I, what have you found to be particularly difficult/challenging for students when working with optimization problems? Explain.
5. What are some of the effective ways you have used to help students overcome challenging/difficult steps involved in solving optimization problems in your Calculus I classes? Explain.
6. What teaching technologies, if any, do you use to support students' learning about optimization derivatives in your Calculus I classes? How do you use these technologies? Explain.
7. What other tool(s), besides technology, do you use to support students' learning about optimization problems in your Calculus I classes? How do you use this tool(s)? Explain.
8. What real-world contexts (e.g., economics), if any, do you typically use in your teaching of optimization problems in your Calculus I classes? Explain.
9. Please state the title, author (s), and edition (if applicable) of the calculus textbook you have used in your recent teaching of Calculus I.
10. What do you consider to be a weakness(es), if any, in how Calculus I textbooks you have used present optimization problems (e.g., in expository sections, examples, or exercises, respectively)? Explain.
11. What do you consider to be a strength(s), if any, in how Calculus I textbooks you have used present optimization problems (e.g., in expository sections, examples, or exercises, respectively)? Explain.

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**Appendix B-Effective strategies for teaching optimization problems in calculus**

Table 1. Experts' responses to Item 5 in the questionnaire

| <b>Expert</b> | <b>Response to Item 5-</b> What are some of the effective ways you have used to help students overcome challenging/difficult steps involved in solving optimization problems in your Calculus I classes? Explain.  |
|---------------|--|
| E1            | We often talk through the big ideas (generally) without doing all of the steps. I also give them several problems to solve (many of which are graded).   |
| E2            | I give them a step by step process, that involves writing the objective function (including realizing that they are wrong if their function has more than one variable), defining the feasible interval, finding critical points, and proceeding from there.   |
| E3            | I try to provide step-by-step guidelines to break the problem into manageable steps.   |
| E4            | I always tell students to draw a diagram of the object/scenario they wish to optimize and label appropriate quantities. This can help them to realize the formula for the appropriate geometric shape.   |
| E5            | For interpreting problems, I have often found it useful to encourage students to draw and label diagrams whenever possible to help them visualize the relationships.   |
| E6            | I always encourage them helping each other, drawing pictures, and leveraging all of their resources that they have online or in-person.  |
| E7            | We focus first on what quantity we are trying to maximize or minimize. We set up an equation to represent that quantity, and then we ask ourselves if the equation we have written down describes our exact problem (for example, if we know our rectangle is supposed to have an area of $400\text{cm}^2$ , then is that information incorporated into our function?) We do several problems where they *only* set up the function. This allows for more practice and experience setting up the function. |
| E8            | It is important to show the students that they are capable of doing these problems on their own. Thus, I tend to give them a lot of "easy"   |

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optimization problems to try on their own first before I move on to more difficult examples. I also give them plenty of opportunities for practice. This includes in-class and outside-of-class worksheets.

- E9 I tell them that the construction of objective function proceeds in two steps: (1) finding a formula for the quantity to be maximized/minimized; often it is a known geometric formula that should be memorized. (2) reducing the number of variables to 1, if there are more than 1. This requires one to express the relation between different variables as an equation, and to use it to eliminate one of the variables.
- E10 Honestly, I do not have any specific effective ways to help students to overcome those steps other than (1) explain why those steps are important and (2) explain with other examples those mistakes could cause wrong answers.
- E11 Examples, group quizzes in class.
- 

### **Appendix C-Additional effective strategies for teaching optimization problems in calculus**

Table 2. Experts' responses to Item 7 in the questionnaire

| <b>Expert</b> | <b>Response to Item 7-What other tool(s), besides technology, do you use to support students' learning about optimization problems in your Calculus I classes? How do you use this tool(s)? Explain.</b>   |
|---------------|--|
| E1            | I provide lots of class time for students to work on problems in groups. I also provide access to several videos (that we made during Covid) that shows the set up and sometimes the full solution to several of these problems. I encourage them to try them on their own before watching the videos. |
| E2            | Nothing I'd call a tool. I'm not sure what a tool would look like for this kind of question. Even though these problems are often visual and physical, I can't think of any way to do them other than to learn to set them up on paper.  |

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- E3 GeoGebra has many visual apps to provide understanding of the problems.
- E4 In-class activities I use are carefully designed to guide them through an example (usual lecture), then they practice a similar example but with the constraint and optimizing function reversed. E.g. I show them how to maximize volume for a fixed surface area, then they are tasked to minimize surface area given a fixed volume.
- E5 Sometimes I use real objects in my classes and have students manipulate them before we move on to Technology tools. Many students care about how the mathematics they are learning can be applied in real life so I find it helpful to demonstrate some of the real-life scenarios in my classroom. The popular oil spill problem for example, can be easily demonstrated in class using a large water container and oils of varying viscosity.
- E6 I haven't really used other tools.
- E7 If the problem involves in geometry, we always draw and label the shape.
- E8 I use in-class and outside-of-class worksheets. We also have recitation where students get extra practice on optimization problems.
- E9 None
- E10 No special tools.
- E11 Gradescope.com is a good technology. It allows graders to send scores more immediately to students and allow students to more directly and freely argue for points back to the specific graders (anonymously).
-